

Yaw Moment Diagram: Destabilizing thoughts

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Introduction

(To whomever is impatient, knows the main classic definitions of stability, or both: you can jump to around page 9, *Milliken Moment Method* section).

Among the infinite “algorithmic” tools designed for engineers, I’ve always been particularly fascinated by graphical solutions/methods. They used to form an important part of an engineer’s daily life, though the prevalence of extremely cheap computers has been rapidly relegating them to the role of rarities, odd animals on the brink of extinction. Mohr circles, but also temperature-enthalpy diagrams, things that, while ridiculously imprecise compared to numerical methods, give you an intuitive understanding of a physical phenomenon without having to input a number or perform calculations.

There is a number of such things in Vehicle Dynamics: handling diagrams, GG plots, even some classic Ackermann geometry “calculators”. But I think that the title of king of this genre belongs to the Yaw Moment Diagram, or Force-Moment Diagram, and relative method.

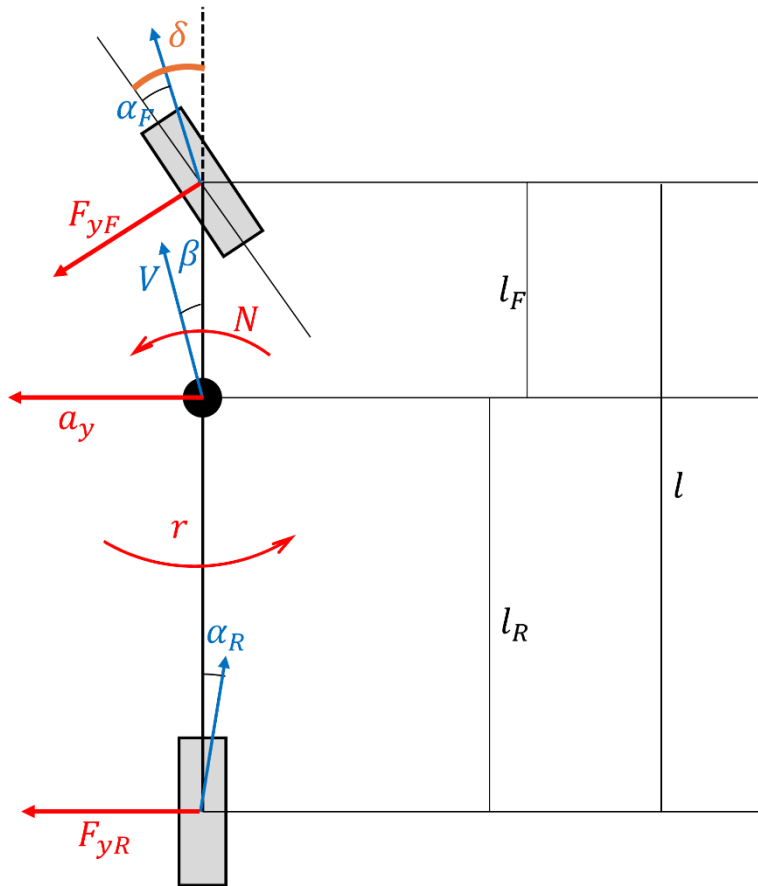
Invented by MRA, the Moment Method promises something that any vehicle dynamics engineer would treasure: a comprehensive, intuitive “picture” of a car behavior along a wide variety of operating conditions, easily explorable on a screen or a piece of paper, with no need to run the immense variety of simulated use-cases, objective maneuvers, subjective evaluations that we normally do.

Determined to understand it better than the “intuitive” idea of a weathercock, I tried to build (by extensively stealing from Pacejka and Milliken) some algebraic framework to understand what it’s made of, how does it work mathematically.

If it works, something I have more doubts about than when I started looking into it.

What follows is the description of that basic algebraic framework, an incomplete review of some different criteria to define stability, and finally, a comparison between the graphical solution to the stability problem proposed by OptimumG, an extension of Milliken’s Stability Index, and a more rigorous (but unluckily, not graphical) approach based on linear system theory.

The bicycle model



Symbol	Description
m	Mass
J	Yaw inertia
N	Yaw moment
a_y	Lateral acceleration
r	Yaw rate
δ	FW steer angle
β	Body sideslip
V	Vehicle speed
V_y	Lateral speed (vehicle reference)
V_x	Longitudinal speed (vehicle reference)
l_F	Front axle to CoG
l_R	Rear axle to CoG
l	Wheelbase
α_F	FW slip angle
α_R	RW slip angle
F_{yF}	FW lateral force
F_{yR}	RW lateral force

A good starting point to understand different stability definitions and metrics is the good, old bicycle model, whose main parameters and sign conventions are reported above (all quantities represented positive). The model is comprised by a car body characterized by a mass and a rotational inertia (m , J), and it's completely defined, from a geometric standpoint, by two lengths, l_F and l_R , representing the distance between the CoG and the front axle, and the CoG and the rear axle; the sum of the two distances gives us the wheelbase l .

The rear axle is described by a tire which remains always aligned with the body (we'll keep it simple, no rear-wheel-steer). The angle connecting the direction the rear tire is pointing to the speed vector of the rear tire center is called α_R (rear slip angle), defined positive when the tire is rotated counterclockwise with respect to the speed vector. The rear tire produces a lateral force which is a function of (at least) the rear slip angle, F_{yR} .

The front axle follows the same definitions and sign convention as the rear one, with the added complexity that it rotates with respect to the body by an angle δ . As the rear one, it produces a lateral force F_{yF} positive when pointing to the left side.

The vehicle's movement is described by the speeds and accelerations at its center of gravity, which we'll call V_x and V_y . These speeds are expressed in the vehicle's reference frame: V_x describes the speed component parallel to the direction the body is pointing to, while V_y indicates the lateral speed, positive towards the left. The vehicle's rotational speed is called " r " as yaw *rate*, and it's positive for a counterclockwise velocity.

It's usually more comfortable (especially when linearizing equations) to replace the lateral speed of the vehicle V_y with the body sideslip angle β , defined as:

$$\beta = \tan^{-1} \left(\frac{V_y}{V_x} \right)$$

Kinematically, we can hence express the relations between slip angles and vehicle motions as:

$$\alpha_R = -\tan^{-1} \left(\frac{V_y - r * l_R}{V_x} \right)$$

$$\alpha_F = \delta - \tan^{-1} \left(\frac{V_y + r * l_F}{V_x} \right)$$

Dynamically, we'll impose that the vehicle must find itself in equilibrium with respect to lateral displacement and yaw rotation, that is:

$$m * a_y = F_{yF}(\alpha_F) * \cos \delta + F_{yR}(\alpha_R)$$

$$N = Jr' = F_{yF}(\alpha_F) * l_F * \cos \delta - F_{yR}(\alpha_R) * l_R$$

Provided that we know the constitutive relation between lateral forces and slip angles (plus whatever element we want to complicate it with), we can arrange the above equations in an ordinary differential equation form which we'll be able to integrate in time; however, it's much more fun not to do so, and instead further simplify the model, in such a way that it becomes trivial to understand.

The kinematically-linearized bicycle model

If we assume small angular displacements, we can rewrite the kinematic relations between the above variables as:

$$\beta = \frac{V_y}{V_x}$$

$$\alpha_R = -\beta + \frac{l_R}{V_x} * r$$

$$\alpha_F = \delta - \beta - \frac{l_F}{V_x} * r$$

We'll also apply the "small displacements" simplification to the equilibrium equations, which will lead us to neglect the $\cos(\delta)$ component and write them in the simpler form:

$$m * a_y = F_{yF}(\alpha_F) + F_{yR}(\alpha_R)$$

$$N = Jr' = F_{yF}(\alpha_F) * l_F - F_{yR}(\alpha_R) * l_R$$

Note that there is no specific limitation about the equilibrium condition we consider these variables to be *smaller displacements* from: while most often used to explain the behavior of a vehicle in close-to-straight-line operating condition, we can use it just as well to describe small displacements with respect to a larger-displacement, known operating condition of the vehicle.

This simplified version of the bicycle model does not involve any specific assumption about the behavior of the tire-axles, which are still generic functions of their respective slip angles. In fact, the

“small displacement” approximation is still acceptably accurate for the relatively large angles required to reach limit lateral acceleration in dry asphalt conditions.

We will stop here (i.e., we will keep nonlinear tires, but linearize kinematic relations) to discuss some of the stability criteria we’ll go through; however, in order to reach a model which we can *fully* understand, we need to go a step further and perform another sacrifice, and solemnly renounce to non-linear tire force-slip angle relations.

Linear tires bicycle model (and some other nice simplifications)

By the slightly misleading expression “linear tire”, we mean a tire whose force is linearly related to slip angle, i.e.

$$F_{yi} = C_i * \alpha_i$$

If we extend the concept to small perturbations around an equilibrium point, we can define C as the local derivative of the force-slip angle curve.

$$C_i := \left. \frac{\partial F_{yi}}{\partial \alpha_i} \right|_{\alpha_{i,equl}}$$

Now, it would be more rigorous to denote small displacement with some symbol such as ∂ , however, for the sake of readability, we will use the same formulation for both a completely linearized bicycle model (where the concept of small variations and totals is interchangeable) or for a linearized model describing smaller perturbations around a previously-defined state.

$$F_{yF} = C_F * \alpha_F$$

$$F_{yR} = C_R * \alpha_R$$

The equations we found before can therefore be further simplified into:

$$m * a_y = C_F \alpha_F + C_R \alpha_R$$

$$N = Jr' = C_F \alpha_F l_F - C_R \alpha_R l_R$$

Which already becomes far more understandable by exposing the relation between lateral forces and moments and slip angles. Depending on what we are interested into, we can either use it directly, or introduce a variable substitution, by replacing α_F and α_R with δ and β terms. It’s particularly practical to do so after introducing the following quantities (from [1] Chapter 1):

$$C = C_F + C_R$$

$$Cs = C_F l_F - C_R l_R$$

$$Cq^2 = C_F l_F^2 + C_R l_R^2$$

The term “ q ” is mostly there for brevity and has no intuitive physical meaning (at least, that I understand) apart from the useful reminder, being squared, that it is always greater than zero. Term C represents the sum of the two cornering stiffnesses and it might be seen as a sort of “equivalent stiffness” which is resisting forces “pushing” the vehicle laterally.

Most interestingly, “ s ” is the distance between the CoG and the neutral steering point, positive when the latter is located ahead of the former.

Since the neutral steering point can be defined as the point where the application of a force only results in an equal slip at the front and rear tires:

- If the CoG is located *behind* this point ($s > 0$), centrifugal force causes the rear axle to slip more than the front one, leading to oversteer.
- If the CoG is located *in front of* this point ($s < 0$), centrifugal force causes the front axle to slip more than the rear one, leading to understeer
- If $s = 0$, the vehicle is neutral: any lateral acceleration results in an equal slip angle at both axles.

Hence, if we rewrite the lateral force and moment balances using these artificial terms, we obtain a quick representation of the dynamic behavior of the vehicle which already includes, built-in, an indication of under/oversteer:

$$m * a_y = C_F \delta - C \beta - C s \frac{r}{V_x}$$

$$N = J r' = C_F l_F \delta - C s \beta - C q^2 \frac{r}{V_x}$$

This system of equations will be the foundation of many of the stability-related considerations that'll follow.

Steady-state behavior

Before finally immersing ourselves in the world of stability, it is useful to characterize the model in its steady-state form, by solving the above equations for $r' = 0$, that is, imposing that yaw rate is constant in time, which implies steady-state conditions.

By manipulating the above equations, we obtain the following two expressions:

$$\delta = \frac{l r}{V_x} - \frac{m C s}{C_F C_R l} a_y$$

or alternatively:

$$\delta = \frac{l r}{V_x} + m \left(\frac{l_R}{l C_F} - \frac{l_F}{l C_R} \right) a_y$$

and:

$$\beta = \frac{l_R r}{V_x} - \frac{m l_F}{C_R l} a_y$$

In which we see that, under the steady-state assumption, both front wheel steer angle and sideslip can be expressed as the sum of a kinematic contribution (depending on yaw rate and speed) and of a dynamic contribution, depending on lateral acceleration multiplied by some quantity involving:

- For sideslip, the static mass acting onto the rear axle, divided by the rear axle cornering stiffness, or the negative product of:
 - Fraction of lateral force to be exerted by the rear axle
 - Rear cornering compliance
- For steered wheel angle, the product of:
 - Total lateral force

- Stability Margin (as defined in [2], Chapter 5, $-s/l$)
- Sum of front and rear cornering compliance ($\frac{c}{C_F C_R}$)
- Or, alternatively:
 - Total lateral force
 - Difference between:
 - Front lateral force multiplied by front cornering compliance
 - Rear lateral force multiplied by rear cornering compliance

There are, really, infinite ways to *visualize* and to decompose components of the above relations; they become particularly meaningful when we ask ourselves what happens when we slowly move through steady-state condition by keeping either cornering radius or vehicle speed constant. In both cases, we should get rid of the “ r ” term by showing its dependency on the variable we are imposing as a constant. Which leads to:

- Constant speed: $\frac{r}{V_x} = \frac{a_y}{V_x^2}$
- Constant radius: $\frac{r}{V_x} = \frac{1}{R}$

Where R is the radius of the corner the vehicle is negotiating. The following relations ensue:

Constant Speed:

$$\delta = \left(\frac{l}{V_x^2} - \frac{mCs}{C_F C_R l} \right) a_y$$

$$\beta = \left(\frac{l_R}{V_x^2} - \frac{ml_F}{C_R l} \right) a_y$$

$$\frac{\partial \delta}{\partial a_y} = \frac{l}{V_x^2} - \frac{mCs}{C_F C_R l}$$

$$\frac{\partial \beta}{\partial a_y} = \frac{l_R}{V_x^2} - \frac{ml_F}{C_R l}$$

Constant Radius:

$$\delta = \frac{l}{R} - \frac{mCs}{C_F C_R l} a_y$$

$$\beta = \frac{l_R}{R} - \frac{ml_F}{C_R l} a_y$$

$$\frac{\partial \delta}{\partial a_y} = -\frac{mCs}{C_F C_R l}$$

$$\frac{\partial \beta}{\partial a_y} = -\frac{ml_F}{C_R l}$$

By comparing the two formulations, one should notice the fact that CR expressions are “cleaner”, that is they do not carry any dependency with speed (which is, however, implicit in the fact we are, indeed, increasing speed in order to achieve higher lateral acceleration levels).

Furthermore, in a constant-radius maneuver:

- The rate of change of β with respect to lateral acceleration is always negative (for a positive C_R)
- The sign of the rate of change of δ with respect to lateral acceleration, if both (C_F and C_R are positive) only depends on the stability margin: positive for understeer, negative for oversteer. This quantity can be defined as the *understeer gradient* ([1], Chapter 1; [2], Chapter 5).

Instead, during a constant-velocity (ramp steer) maneuver we see that:

- Sideslip might either increase or decrease with lateral acceleration depending on speed. At lower speeds, it is expected to be positive.
- For steered wheel angle, the upwards or downwards trend of delta with speed, “instinctively associated” with US/OS balance, is partly *masked* by kinematic effects, which introduce an always positive, and greater for lower speeds, contribution.

These differences may seem obvious (they are), however it is common to sometimes get confused about the meaning of body sideslip gradient, steering wheel angle gradient and so on, mixing concepts from incompatible maneuvers.

An interesting consequence of our ability to find the expected values and gradients of steered angle with speed leads us to a first family of criteria to define vehicle stability.

Steady-state based criteria

Steady state-based criteria are a family of criteria to determine stability based on the observation of the states of the vehicle in either steady-state conditions or a sequence thereof (for example, a sequence of steady-state equilibria in which we are observing state variables for different lateral acceleration values).

I particularly like them, because they are extremely simple in their definition, and they’re based on the description of a situation which is somewhat relatable and intuitively understandable, which does not necessarily apply to some of the more robust ones.

Constant Radius

For a constant-radius maneuver, a car can be defined stable if the front steered angle needed to negotiate a corner is above zero. This roughly describes the case of driving a car along a steering pad: in case the vehicle is oversteering, with an increase in speed we will have to progressively reduce the steering angle in order to maintain the vehicle on the intended trajectory, up to a point where we’ll need to *countersteer*, i.e. turn the wheel in the opposite direction of the corner.

It may seem somewhat arbitrary to define a null front wheel steer angle as the threshold for instability: as anyone having driven an oversteering car may know, it is indeed possible to keep a car in equilibrium with a front steered angle below zero for reasonably high amounts of time; the consequences of the criterion become somewhat clearer when analyzing it in yaw rate.

Anyhow, by simply taking the constant-radius expression for *delta* we found above, we’ll have

$$\delta = \frac{l}{R} - \frac{mCs}{C_F C_R l} a_y$$

When turning to the left, with l , R , m and all cornering stiffness terms positive (which is a reasonable assumption only far from limit conditions), the sign of δ only depends on the sign of s and a_y , namely:

- If $s < 0$ (understeer), δ is positive for all a_y values (provided C , C_F , C_R are all positive)
- If $s > 0$ (oversteer), there will exist a limit lateral acceleration condition for which the right expression in the equation will reach zero.

This second condition is commonly translated from a lateral acceleration one into a longitudinal speed condition by imposing steady-state to find:

$$0 = \delta = \frac{l}{R} - \frac{mCs}{C_F C_R l} \frac{V_x^2}{R}$$

$$V_{cr} = \sqrt{\frac{C_F C_R l^2}{mCs}}$$

This speed value, above which the vehicle will be unstable, is called *critical speed*. It's clear that, under *normal* conditions, it is only defined for a positive “ s ”. For understeering vehicles, it is instead possible to define a similar “characteristic speed”, which identifies speed at which the yaw rate response hits a maximum, which, however, falls out of the realm of stability.

Constant Speed

The steady-state, constant-velocity criterion is, in my opinion, even more intuitive. It says that the vehicle becomes unstable when the derivative of δ with respect to a_y falls below zero. This is immediately apparent: considering low slip angle conditions, we expect the vehicle to turn in the direction of the steering wheel rotation; if equilibrium is only found for a counter-steer condition, it is clear that, at a given speed, we won't be able to reach that by slowly moving through equilibria.

In practical terms, the equation we found before for $\frac{\partial \delta}{\partial a_y}$ (constant V_x) yields almost exactly the same result as the previous one when imposing it equal to zero.

$$\frac{\partial \delta}{\partial a_y} = \frac{l}{V_x^2} - \frac{mCs}{C_F C_R l} = 0$$

Which, again, yields:

$$V_x = V_{cr} = \sqrt{\frac{C_F C_R l^2}{mCs}}$$

For an oversteering vehicle, there will exist a speed in which the gradient of δ with respect to a_y will become negative, thus implying instability.

Two small remarks here:

- The above analysis is based on the assumption that the behavior of the axles does not depend on speed. This is reasonable for road vehicles which do not generate strong aerodynamic forces, much less for racecars whose balance may change depending on speed. The same holds for whatever other system whose behavior is speed-dependent and affects the way tires generate forces.
- The second criteria is more relevant for nonlinear tire behavior. In fact, it is still possible to apply it with non-linear tires by observing the evolution of the front steered angle with lateral

acceleration; this comes with the advantage that, being evaluated at constant speed, we can indeed apply it for vehicles equipped with aero or other speed-dependent devices.

A good way to visualize the second criterion for a non-linear model is through a *handling diagram*. In it, we use the left axis to plot the difference between front and rear tire slip angles versus lateral acceleration. We then take the right part of the plot to show how the variable l/R increases with a_y (linearly with speed).

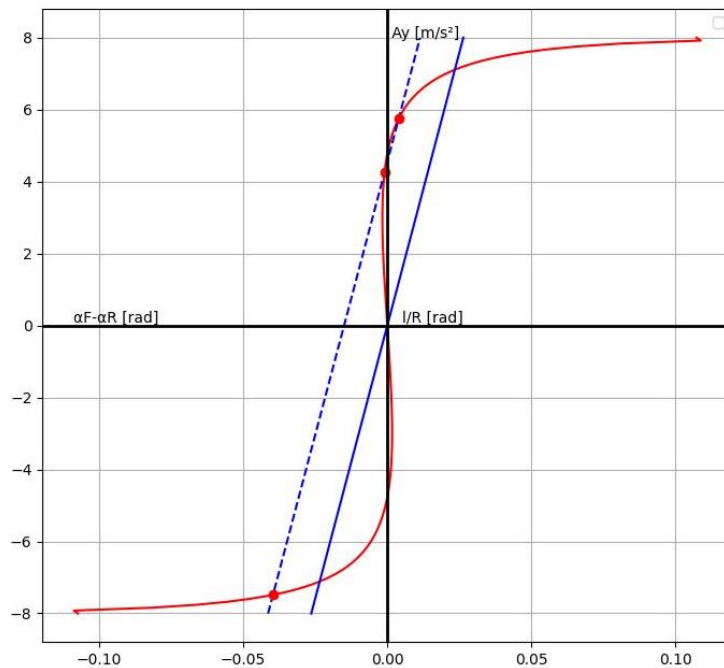
We know, from the equilibrium equation, that:

$$\delta = \frac{lr}{V_x} + m \left(\frac{l_R}{lC_F} - \frac{l_F}{lC_R} \right) a_y$$

Or

$$\delta = \frac{l}{R} + \alpha_F - \alpha_R$$

With this, we can graphically find delta as the sum of the two contribution from left and right parts of the plot. It's clear that, when the left part becomes parallel to the right one, the derivative of delta with lateral acceleration becomes zero, which means we reached an instability condition.



In the above plot, representing an oversteering vehicle, the $\alpha_F - \alpha_R$ curve first rises, then decreases. The solid blue line, identifying a given constant speed maneuver, assumes a more vertical slope for increasing speed values. This progressively reduces the value of a_y that we can reach in stable conditions, up to a speed (critical speed) where the vehicle becomes unstable for any value of lateral acceleration.

By drawing a parallel line (dashes) to the one identifying speed, and finding where it crosses the $\alpha_F - \alpha_R$ curve, we can identify equilibria (both stable and unstable) for a given speed and front wheel steer. In the above example, three equilibrium points exist: a first one at high, negative acceleration (unstable), a second one at $\sim 4 \text{ m/s}^2$ (stable), a third one at $\sim 6 \text{ m/s}^2$ (unstable). The stability of such points is determined by whether the slope of the red curve (in the YX plot) is higher (stable) or lower (unstable) than the dashed line.

Yaw Rate ODE formulation-based criterion

Another widespread criterion to judge on the stability of a linearized bicycle model is based on synthesizing the equilibrium equations into an ODE form, by replacing lateral acceleration in lateral forces equilibrium with the multiplication of longitudinal speed and the sum of yaw rate and sideslip derivative. The second equation (Z momentum equilibrium) is then derivated in time, and body sideslip derivative can be replaced in either equation to write a differential equation in a single variable (yaw rate r). Its formulation, which I'm borrowing from Pacejka ([1], again, Chapter 1), is reported below:

$$mJV_x^2 * r'' + mCV_x(q^2 + k^2) * r' + (C_F C_R l^2 - mCsV_x^2) * r = ml_F C_F V_x^2 * \delta' + C_F C_R V l * \delta$$

The relevance of this formulation is immediately apparent due to the similarity of this expression to that of a second-order mechanical system, i.e., for example, a pendulum (as explained in [2] Chapter 6). Stability can then be analyzed from a classical vibrational analysis standpoint, by identifying root loci.

By rewriting the homogeneous part of the equation and simplifying terms as it follows (using, again, a notation taken from Pacejka's introductory chapter),

$$a_0 * r'' + a_1 * r' + a_2 * r = 0$$

we find that the two roots are located by:

$$\lambda_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0 a_2}}{2a_0}$$

As argued by Pacejka:

- The a_0 component is always positive
- The a_1 component is always positive for a vehicle linearized in the vicinity of null steering angle and null sideslip, though it might get negative if coefficients come from the linearization of a vehicle in close-to-limit conditions (where, for example, the derivative of lateral force with respect to slip angle is becoming negative, leading to a negative C). In this second case, the vehicle will always be unstable since at least one of its roots will have a positive real component.

In case a_1 is positive, the only possibility to find instability comes from the $\sqrt{a_1^2 - 4a_0 a_2}$ part of the equation. Most obviously, if this component yields a real value of greater amplitude than a_1 , one of the two roots will have a positive real part, again, meaning that the system is unstable.

This will happen when $4a_0 a_2$ is less than zero, that is (since mJV_x^2 is always positive), when:

$$a_2 = C_F C_R l^2 - mCsV_x^2 < 0$$

Again, focusing on the case where both C_F and C_R are positive, this term can only become negative when $s > 0$ (oversteering vehicle); solving in speed, we obtain once again the expression for critical speed V_{cr} .

$$V_x > V_{cr} = \sqrt{\frac{C_F C_R l^2}{mCs}}$$

So, in the realm of $C_R > 0$, $C_F > 0$ cases, we found once again that instability only occurs for oversteering vehicles above critical speed.

An interesting aspect to underline here, is that the $\sqrt{a_1^2 - 4a_0a_2}$ term:

- Increases with speed for an oversteering vehicle
- Decreases with speed for an understeering vehicles

This causes a rather significant difference in the nature of roots for the two cases:

- For an oversteering vehicle, roots will generally be real, and the system will *usually* behave as an over-damped system, reaching the steady-state value of yaw rate with no overshoots.
- For an understeering vehicle, roots will start from a real value and then the value under square root will decrease until it falls below zero, leading to the appearance of an oscillatory behavior: the vehicle will overshoot its steady-state yaw rate value.

Summarizing, through the r ODE formulation, we can define a stability criterion which is stated as two necessary and sufficient conditions:

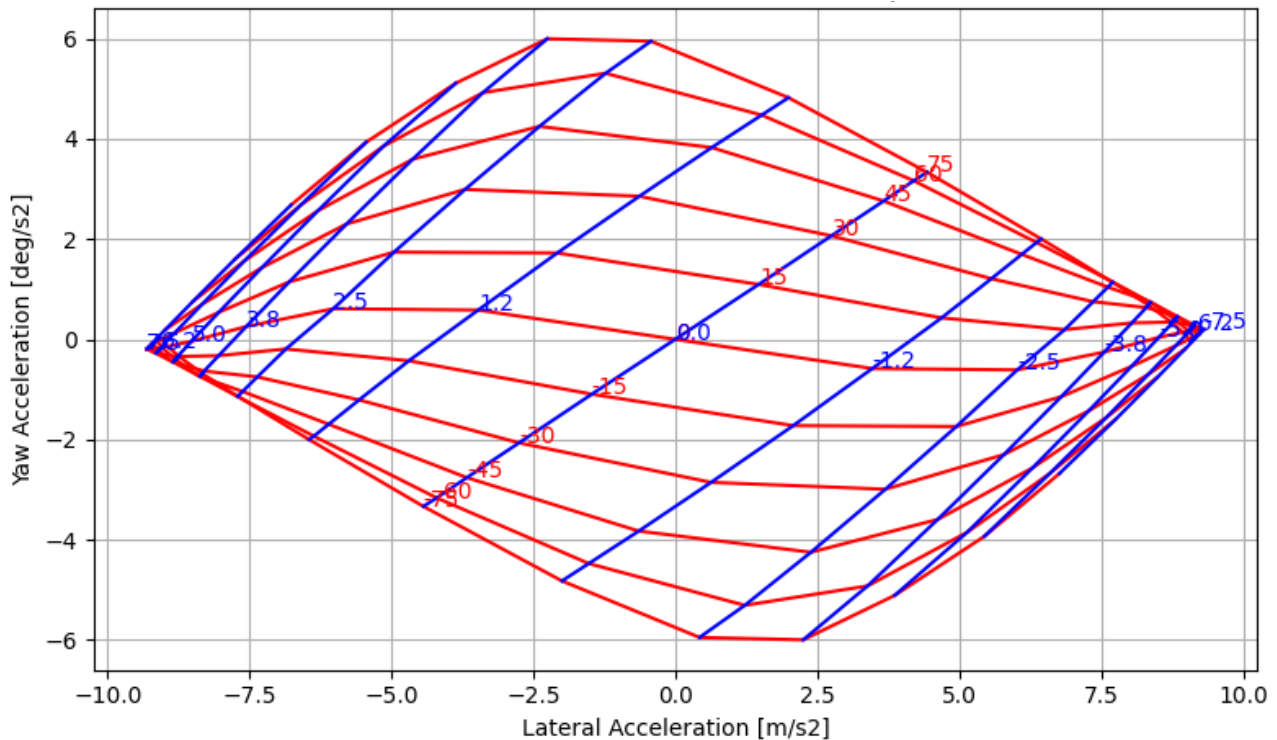
- $C > 0$
- $4a_0a_2 > 0 \rightarrow a_2 > 0 \rightarrow C_F C_R l^2 - m C_s V_x^2 > 0$

In my humble opinion, this is a rather strong criterion, since it directly descends from the well-tested concept of Linear Stability, instead of somehow “reinventing” a definition. We already saw that, for $C_R > 0$, $C_F > 0$, all criteria tell us the same thing: an understeering vehicle is stable at all speeds, while an oversteering one is only stable below its critical speed.

In practice, however, a vehicle might be driven in conditions whether one or both of its axles are brought past their friction peaks, leading to negative values of the related force – slip derivatives. This is especially relevant for methodologies that aim at describing a “generalized” stability criterion. Perhaps the most famous of such approaches is Milliken Moment Method.

Milliken Moment Method

After this long introduction, let's finally come to the analysis which started all of this headache, and whose counter-intuitiveness I'm still fighting by filling numerous notebook pages. With Milliken Moment Method, or Yaw Moment Diagram (MMM, YMD), the vehicle's behavior is analyzed at multiple equilibrium points obtained by imposing a sideslip, a front steered angle and some kinematic “rule” to determine yaw rate. By repeating the operation for multiple sideslip and delta values, one obtains the arguably nice plot reported below.



While not normally used in production vehicles design, the MMM seems to have enjoyed prolonged popularity at least among FSAE teams, perhaps due to it being among the favorite questions of some judges, or thanks to the commercial availability of courses about that, two phenomena not necessarily unrelated.

What, precisely, each point represents is still the subject of many headaches to me, however, the most fitting definition I could think of is, every point where two lines cross represents a state of equilibrium of the car, where:

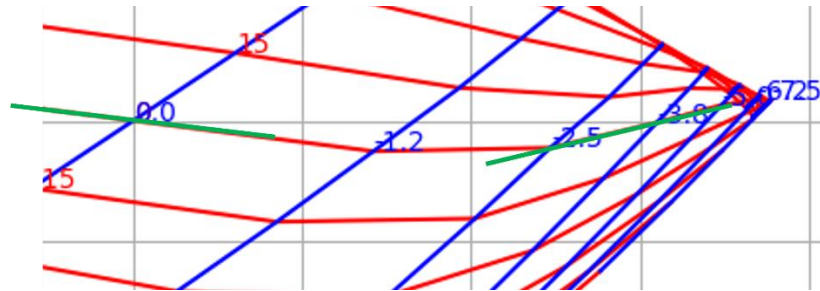
- Steering wheel angle and body sideslip are imposed
- The yaw moment of the vehicle is counteracted by a torque and reported
- Some assumption about the vehicle motion is applied, to be able to determine the unknown yaw rate, namely (either):
 - Constant Speed
 - Constant Radius (among which, as a special case, Infinite Radius)

Milliken goes very deep into detail in Chapter 8 of his classic [2], and I strongly encourage anybody interested to study his presentation of the MMM there to understand its interpretation.

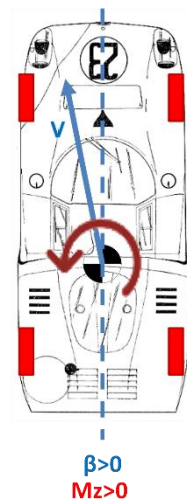
How can MMM help us determine a vehicle's stability? The classic way to understand this is through the “weathercock analogy”, i.e., a stable vehicle moving on a patch of road behaves like a weathercock immersed in moving air. If a perturbation ensues, the weathercock comes back to follow the wind direction: a stable car, when perturbed, “rotates back” to follow the direction of the speed vector.

How does this intuitive explanation translate into a *methodology* to determine if the vehicle, under some specific operating conditions, is stable?

All methods rely (for stability) on identifying an iso-SWA line, and evaluating its slope.



For example, in the above plot (close-up of the one at the beginning of this section), following the red iso-SWA curve at null body sideslip identifies a downward-sloping curve, while performing the same operation when re-crossing the $N = 0$ line at $\sim -3.8^\circ$ finds a positive slope.



In mathematical terms, this equals, graphically, to the estimation of a partial derivative. And about, specifically, what to look for, we can find three lines of thought (for sign conventions, refer to the image above):

- Milliken defines *directional stability* the derivative of “N” (yaw moment) with respect to *beta* under constant-radius conditions. With Milliken’s sign conventions (and with our ones), a positive derivative implies the vehicle is stable.
- Claude Rouelle (OptimumG) further extends the concept to constant-speed YMDs, saying, in his article *The four secrets to chassis happiness* [3], that “*The stability is defined as the change in yaw moment as you vary the chassis slip angle by one degree*”. With our sign conventions, this coincides to saying that the partial derivative of N with respect to sideslip, in constant-velocity conditions, should be greater than zero. We will call this “OptimumG Stability” since it is proposed in promotional material as *the* definition of stability.
- Another definition found in Millikens’ book is that of *stability index*: the derivative, for a vehicle driven at constant speed, of yaw moment N with respect to lateral acceleration. With our sign conventions, the criterion states that such derivative should be negative to ensure stability.

In mathematical terms

Let’s translate these three definitions of stability into mathematical terms, and for a linear vehicle model. Remember that this may also be the outcome of a model linearization well past its linear range: the model is only valid very close to the operating conditions it’s been linearized into.

$$ma_y = C_F \delta - C\beta - Cs \frac{r}{V_x}$$

$$N = Jr' = C_F l_F \delta - Cs\beta - Cq^2 \frac{r}{V_x}$$

Milliken Directional Stability

We already saw that, under constant radius conditions, $\frac{r}{V_x} = \frac{1}{R}$, hence the balance equations become:

$$ma_y = C_F \delta - C\beta - Cs \frac{1}{R}$$

$$N = Jr' = C_F l_F \delta - Cs\beta - Cq^2 \frac{1}{R}$$

Under constant-radius conditions, the system of equation can be immediately solved for any couple of δ, β . There is no inter-connection between the two equations, hence we can find the derivative of N with respect to beta as:

$$\frac{\partial N}{\partial \beta} = -Cs$$

Which is a very robust indication of “balance”: unless C falls below zero (past the limits of both front and rear tyres), it roughly equals the information obtainable with the location of the neutral steering point, or the static margin.

OptimumG Stability

The proposed methodology consists of intercepting the an iso-SWA line at any point of the YMD, and then evaluating the value of N at the closest iso-sideslip interceptions. The intercept with the higher value of sideslip should provide a higher value of N than the intercept with a lower one.

To obtain the equivalent mathematical formulation to this condition, we again use the lateral and yaw balance equations, but here we replace the yaw rate term with the constant-speed assumption, i.e.: $\frac{r}{V_x} = \frac{a_y}{V_x^2}$, leading to:

$$ma_y = C_F \delta - C\beta - Cs \frac{a_y}{V_x^2}$$

$$N = Jr' = C_F l_F \delta - Cs\beta - Cq^2 \frac{a_y}{V_x^2}$$

Which introduce an inter-dependency, through the a_y term, between lateral and moment balances. Since we are using an iso-delta line to find the bond between N and β , we should first obtain a_y from the first equation and then replace it into the second one:

$$\left(m + \frac{Cs}{V_x^2}\right) a_y = C_F \delta - C\beta$$

$$N = \left(C_F l_F - \frac{C_F C q^2}{m C s V_x^2}\right) \delta + \left(\frac{C^2 q^2}{m V_x^2 + Cs} - Cs\right) \beta$$

Now, extracting the partial derivative of N with respect to β , we get:

$$\frac{\partial N}{\partial \beta} = \frac{1}{m V_x^2 + Cs} * (C^2 q^2 - m C s V_x^2 - C^2 s^2)$$

Which, after some clean-up, becomes:

$$\frac{\partial N}{\partial \beta} = \frac{1}{mV_x^2 + Cs} * (C_F C_R l^2 - mCsV_x^2)$$

The OptimumG stability criterion is hence expressed as:

$$\frac{\partial N}{\partial \beta} > 0 \rightarrow \frac{1}{mV_x^2 + Cs} * (C_F C_R l^2 - mCsV_x^2) > 0$$

We can immediately notice a similarity with the Linear Stability criterion, as better summarized later.

Stability Index

This criterion is similar to the former one, in that it also uses constant speed conditions. However, instead of looking for the closest, higher-sideslip and lower-sideslip points, we are just evaluating the slope of the iso-SWA line with lateral acceleration.

I find this easier to use, since one doesn't have to ask himself whether a higher sideslip intersection is found below or above the point of interest: it can just look at the slope of the line: if it's negative, the vehicle should be stable. Starting once again from these two equations,

$$ma_y = C_F \delta - C \beta - Cs \frac{a_y}{V_x^2}$$

$$N = Jr' = C_F l_F \delta - Cs \beta - Cq^2 \frac{a_y}{V_x^2}$$

We will now get β from the first expression and replace what we get into the second one, finding:

$$N = C_F(l_F - s)\delta + \left(ms + \frac{C(s^2 - q^2)}{V_x^2} \right) a_y$$

$$\frac{\partial N}{\partial a_y} = ms + \frac{C(s^2 - q^2)}{V_x^2}$$

By rearranging terms and bringing back $C(s^2 - q^2)$ to its C_F, C_R and dimensional contributions, we get:

$$\frac{\partial N}{\partial a_y} = ms - \frac{C_F C_R l^2}{CV_x^2} < 0$$

Which can be, once again, brought back to a form which is more reminiscent of the Linear Stability Criterion:

$$\frac{\partial N}{\partial a_y} < 0 \rightarrow \frac{1}{C} * (C_F C_R l^2 - mCsV_x^2) > 0$$

Where we multiplied terms by V_x^2 (which, being always positive, does not modify the direction of the inequality) but we kept the inverse of C explicitly stated, since, through sign changes, this term can, indeed, modify the outcome.

Summary

Now, neglecting the *Directional Stability* criterion (which we could define as more of a measure of *balance* than of *stability*), we can compare the two remaining constant-speed criteria with the Linear Stability one in the following table:

Criterion	Linear Stability	OptimumG	Stability Index
YMD procedure	—	$\frac{\partial N}{\partial \beta}$	$\frac{\partial N}{\partial a_y}$
Condition 1:	$\Phi > 0$	$\frac{\Phi}{mV_x^2 + Cs} > 0$	$\frac{\Phi}{C} > 0$
Condition 2:	$C > 0$	—	—

For the sake of compactness, we called

$$\Phi := (C_F C_R l^2 - m C s V_x^2)$$

We can immediately see that the three criteria are very similar. In fact, for an oversteering vehicle with a positive sum of lateral-force-to-slip derivatives (C), they will all output the same result. Both YMD-based methods, however, present some weaknesses.

With the OptimumG criterion, for example, if the $mV_x^2 + Cs$ term falls below zero, the inequality is satisfied by exactly the conditions which get flagged as “instability” in the control-theory formula, potentially leading to false negatives (the criterion flags a stable vehicle as unstable). Furthermore, the absence of an explicit second condition ($C > 0$) exposes the methodology to false positives (the criterion says the vehicle is stable, while it is not) when C falls below zero but Φ remains positive.

For the Stability Index method, things are more subtle. The s term is not an issue here, and its only condition involves both Φ and C , which makes it robust against cases where *either* of them becomes negative. It doesn’t, however, for the egregious case where *both* become negative, and the criterion flags an equilibrium point as stable.

Let’s investigate these “corner cases” closer.

Corner cases

Understeering vehicle at low speed

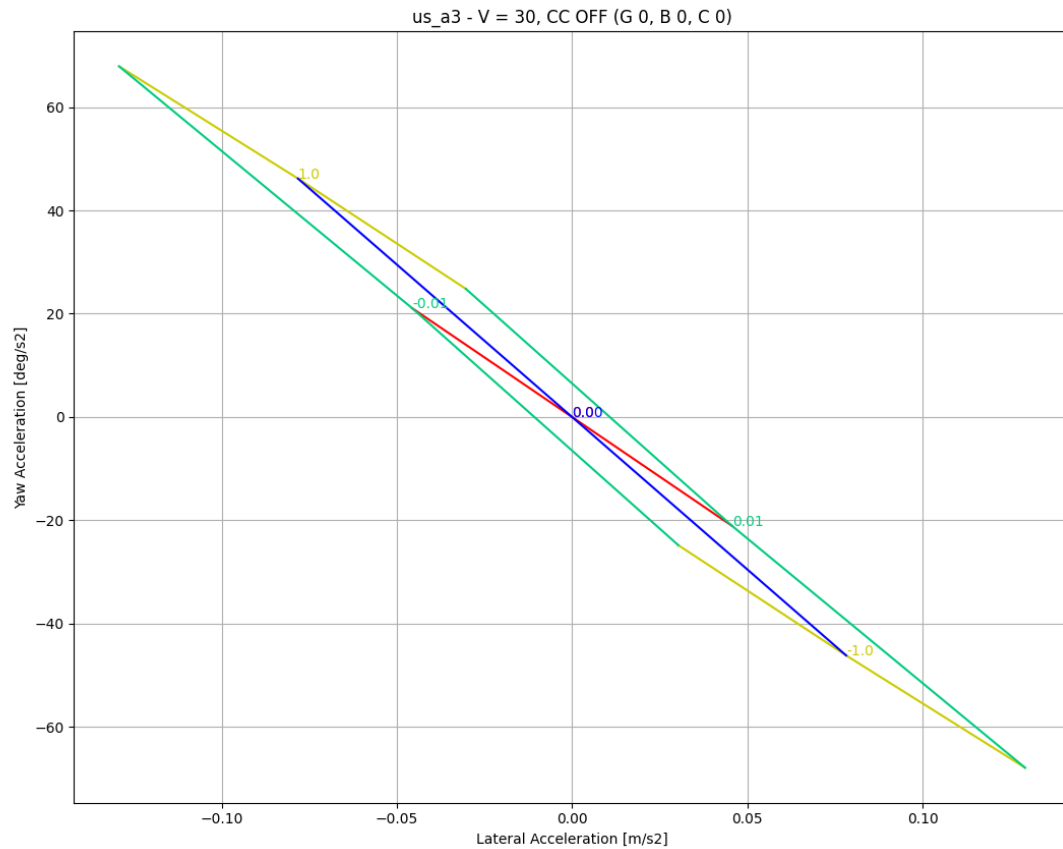
A particularly simple case can be demonstrated by considering the linear model of a vehicle with high cornering stiffnesses and a strong tendency to understeer. Let’s consider the following parameters:

Variable	Value	Unit
m	500	kg
J	720	kg*m ²
l_F	1.2	m
l_R	1.2	m
C_F	103003	N/rad
C_R	199107	N/rad
V_x	8.33	m/s
s	-0.381	m
C	302109	N/rad
$mV_x^2 + Cs$	-80601	Nm
Φ	+1.22e11	

Since Φ is positive and C is positive, the linear stability criterion tells us it’s a stable equilibrium point. But since $mV_x^2 + Cs < 0$ and Φ is positive, their product is negative, thus being flagged unstable by the OptimumG criterion.

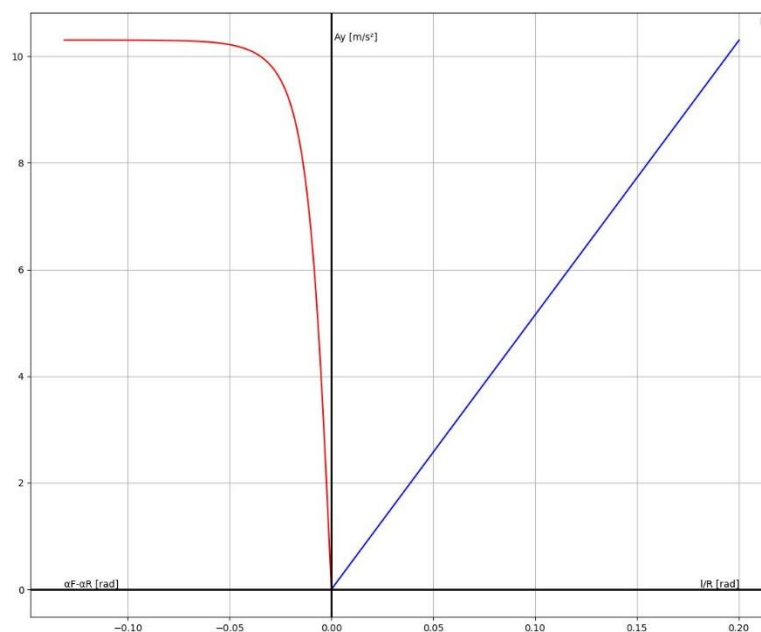
It is quite difficult and not so interesting to draw a YMD at low speeds, which are definitely where this issue appears; however, for very low slip angles and steering wheel values, we get the plot below:

Patrucco, Yaw Moment Diagram: Destabilizing thoughts, 2026



Focusing in zero, and following the red, iso-SWA line towards a higher-sideslip point, and repeating for a lower-sideslip point, we find that the former has a lower YM than the latter. Hence the applied methodology confirms our partial-derivative obtained judgement.

If we look at the Handling Diagram of the (non-linear version) of this vehicle, we get:



That is, the vehicle is stable along its whole working range at 40km/h.

And yes, being honest, I chose a very low mass and unrealistically high cornering stiffnesses to illustrate this concept; however, every understeering vehicle will show this behavior at parking speed (for a road car, below 10 to 30km/h), and by common experience, vehicles are not unstable in those conditions.

For understeering vehicles, we can even define a speed below which the derivative (maneuvering at constant speed) of N with respect to β becomes negative:

$$V_{N\beta} = \sqrt{\frac{-Cs}{m}}$$

In this case, $V_{N\beta}$ equals 15.18m/s, or 54.65km/h. Note that this is **not** the speed at which steady-state body sideslip gradient (at constant speed) changes sign, which tends to be much higher: it's just the speed at which the relation between N/β (with positive C , also a_y/β and a_y/δ) goes from negative to positive.

More interestingly, since the criterion is supposedly valid along the whole diagram, there might be conditions, possibly close or past the limit, where the same situation might occur, leading, again, to a wrong unstable judgement.

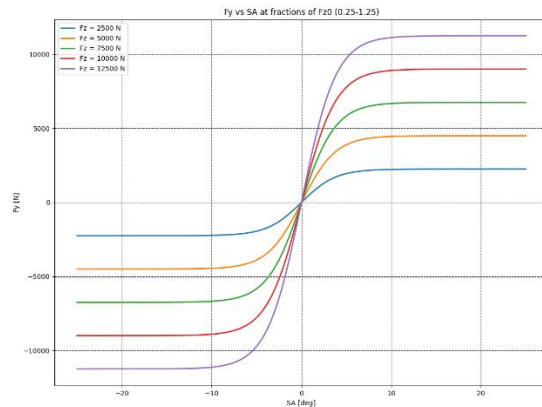
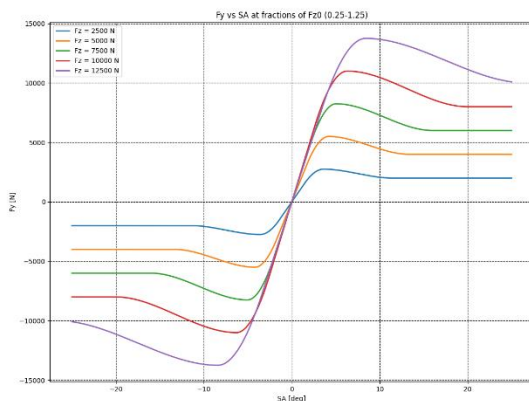
Negative C

While the former case might be considered a bit “academic” (low speed, low mass, extremely high CS), the current one is a bit more concerning.

To prove this, we have to switch to a more involved model, with non-linear tires. Find the its main characteristics below:

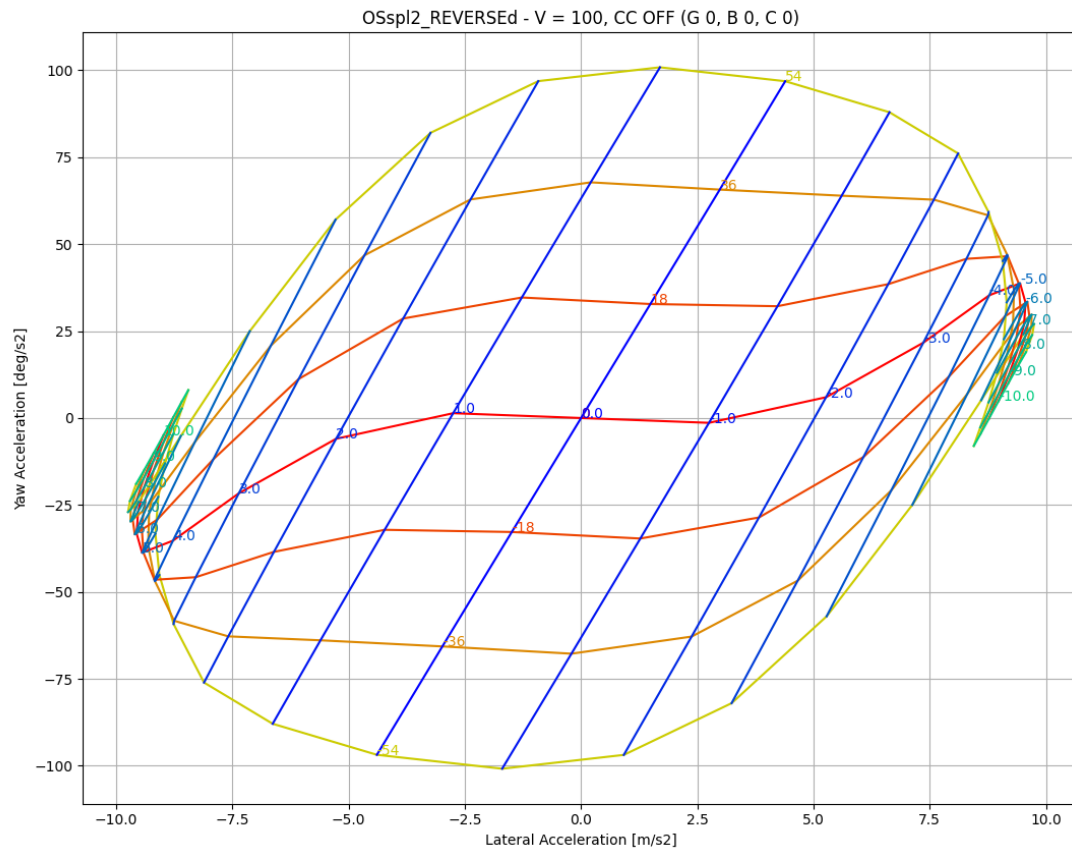
Variable	Value	Unit
m	1000	kg
J	2560	kg*m ²
l_F	1.25	m
l_R	1.25	m

Tires have been defined with two different models (I might talk about in some upcoming article); by now, we can see their F_y - α curves for some fractions of their vertical load here (front, rear):

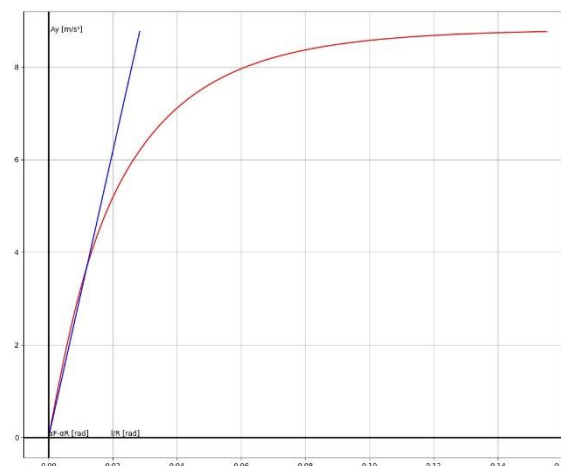


I know that the trend of cornering stiffness with load in the rear tire is unrealistic; however consider that the YMD is obtained at close-to-null longitudinal acceleration and we are not emulating lateral load transfer, so it should not be an issue.

If we plot the YMD at 100km/h, we obtain:

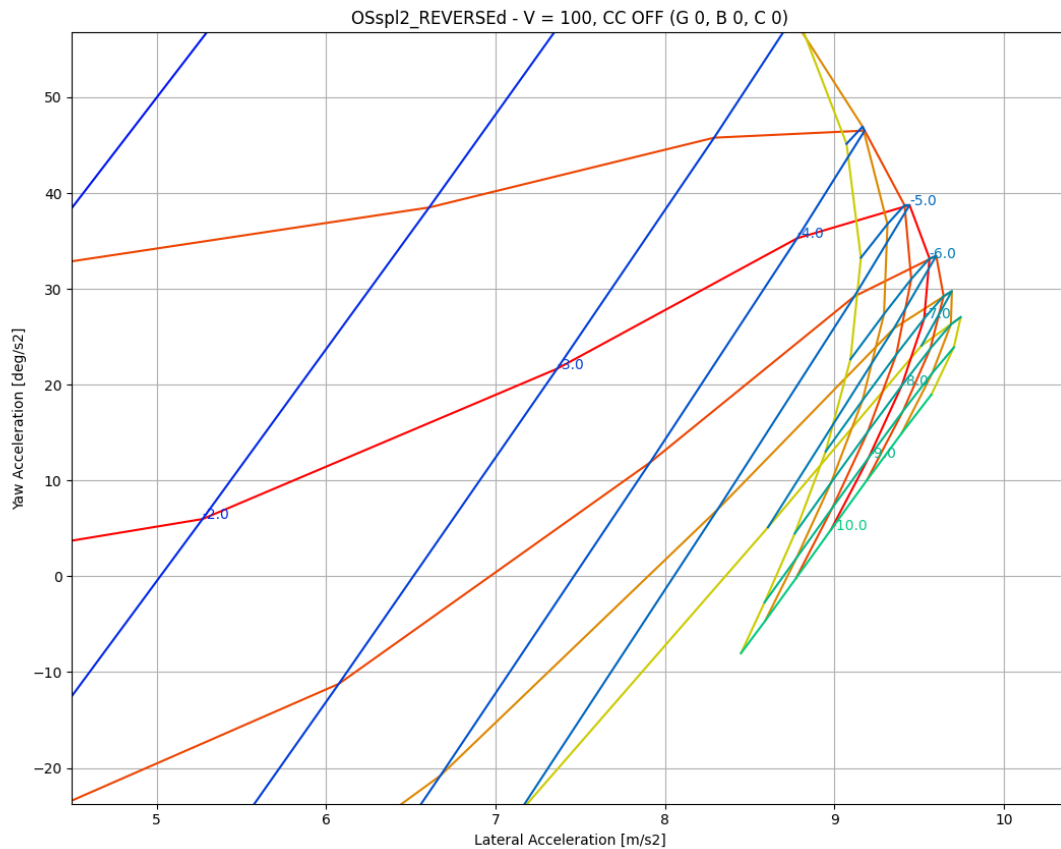


Which clearly tells us that, above 3m/s², our vehicle becomes unstable: take the iso-SWA line (0 deg) between sideslips of -1 and -2 deg: we see that the local derivative $dN/d\beta$ is negative. We could obtain the same information from the handling diagram at the same speed (100km/h):



Patrucco, Yaw Moment Diagram: Destabilizing thoughts, 2026

Here, all criteria agree that at medium lateral accelerations we are far below stable conditions; however, let's see what happens *above* the limit.



In this area of the plot, the vehicle finds another equilibrium point at high sideslip. This “second crossing” of the zero-N line appears close enough to the rear wheel limit that we can still assume that a perturbation might actually be able to make the car “jump” from one to another.

Whatever the reason we got here, let's look at the lines representing sideslips of -10 and -9deg. By following the orange iso-SWA line, we see that the value of N at -9deg is actually *higher* than that at -10deg. Which means that the local derivative $dN/d\beta$ is **positive**. Hence, if we follow the OptimumG criterion, we can conclude that the vehicle is stable.

Except it isn't: if we linearize around this equilibrium point, we get:

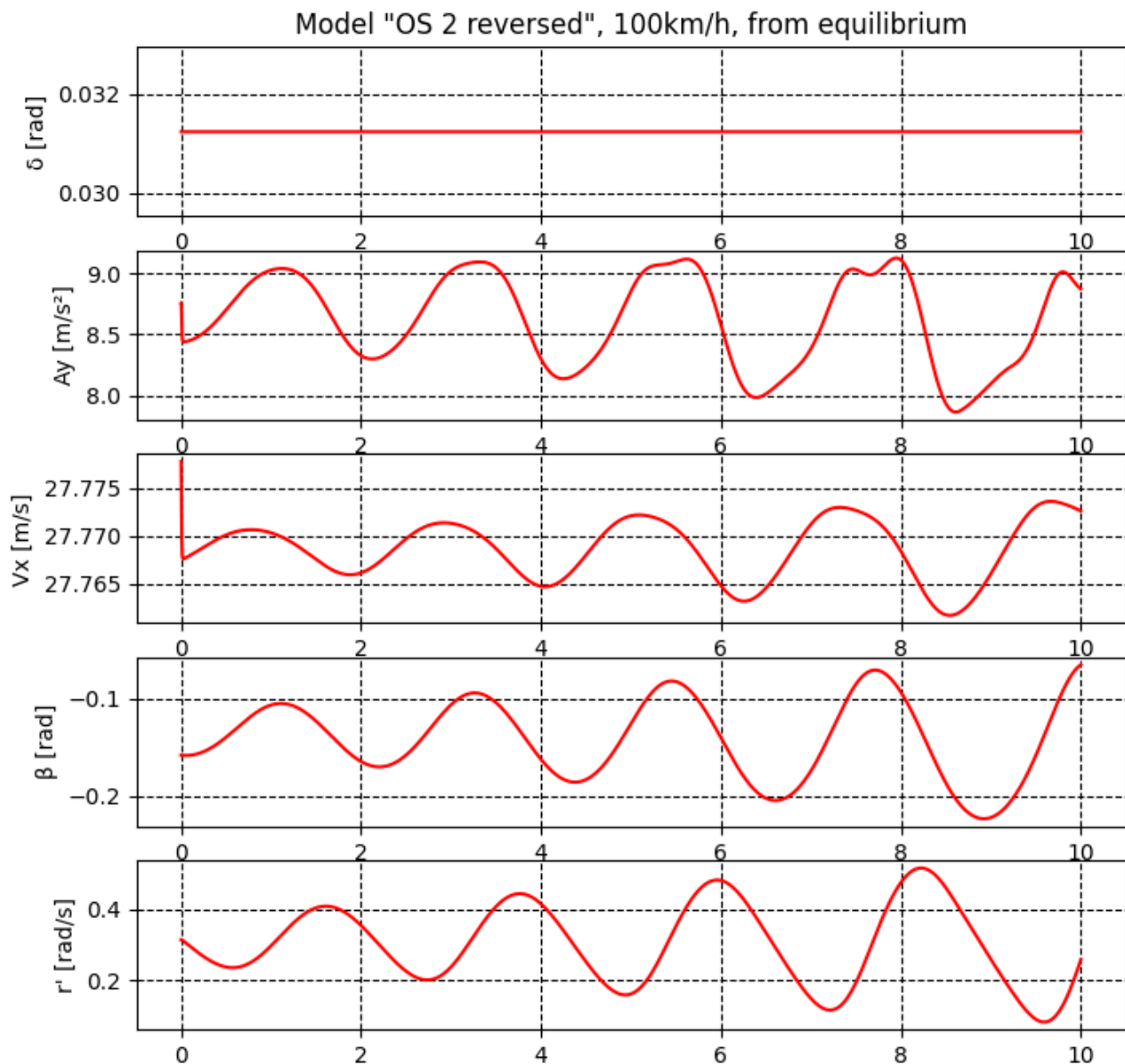
Variable	Value	Unit
m	1000	kg
J	2560	kg*m ²
l_F	1.25	m
l_R	1.25	m
C_F	-12785	N/rad
C_R	1501	N/rad
V_x	27.78	m/s
s	1.583	m
C	-11284	N/rad

$mV_x^2 + Cs$	753748	Nm
Φ	+1.4e10	

Here, what's betraying us is more subtle: the Φ part is positive (which should ensure stability), however C is negative, which would be caught by the control theory formulation, but isn't here.

Due to the high value of V, the $mV_x^2 + Cs$ term is not high enough to produce a sign switch and make the criterion somehow "fix itself": the local $dN/d\beta$ derivative remains positive.

One might argue that, in the end, yaw rate stability is as arbitrary as the weathercock analogy to tell whether a car "is stable"; a good way to understand the implications is to just start from the equilibrium point (plus a small perturbation) and integrate the same model in time to see what happens.



We see how sideslip, yaw rate and lateral acceleration start oscillating with increasing amplitudes, which is a quite fitting definition of instability.

Interestingly, the linearized model did predict this, by finding the following roots:

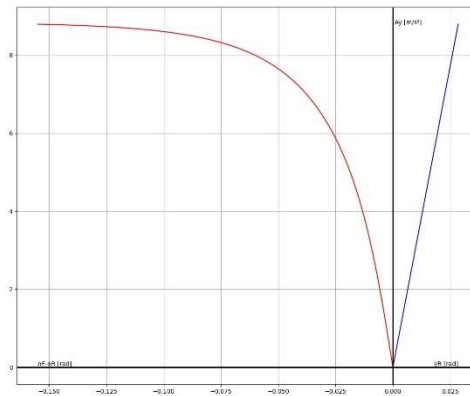
$$\lambda_1 = (0.654 - 5.22j)$$

$$\lambda_2 = (0.654 + 5.22j)$$

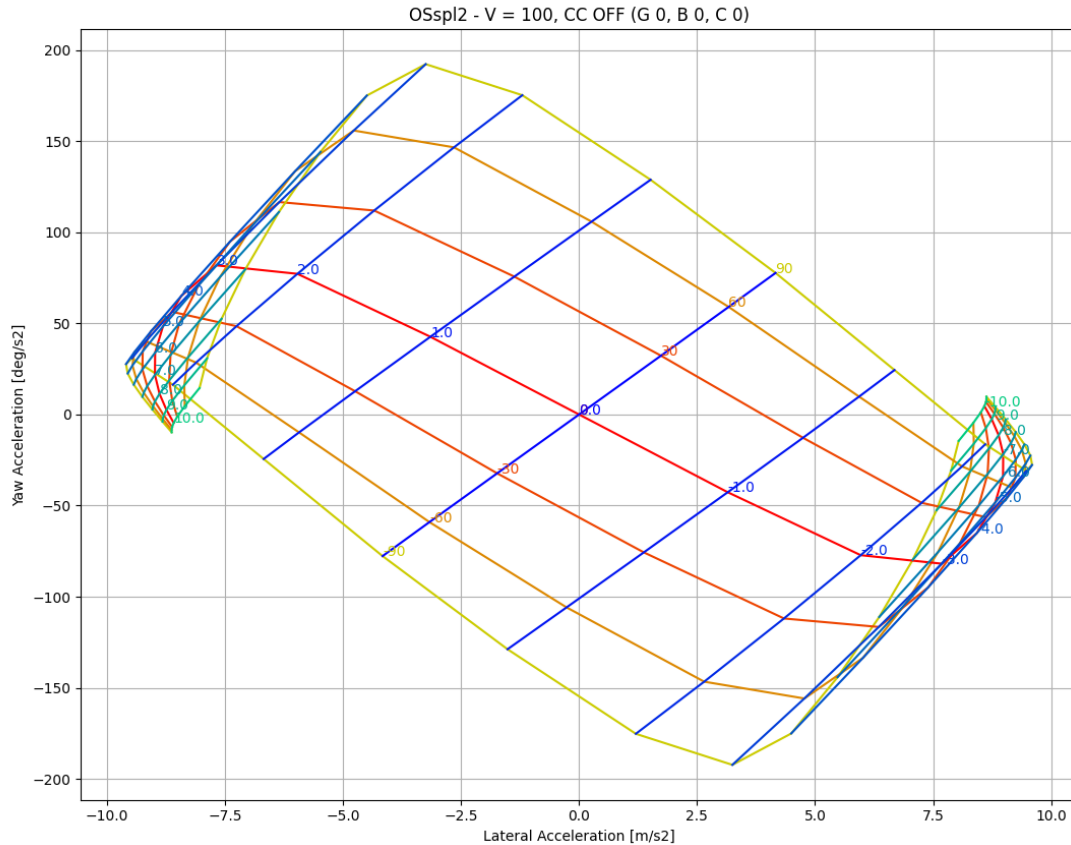
That is, roots have a small positive part (in fact, the increase in amplitudes is quite slow), but a high imaginary part, which shows in the time-integrated plot.

Negative C and Negative Φ

Let's reuse the previous model, but now let's just swap tyres: front model goes to the rear axle, and vice versa.

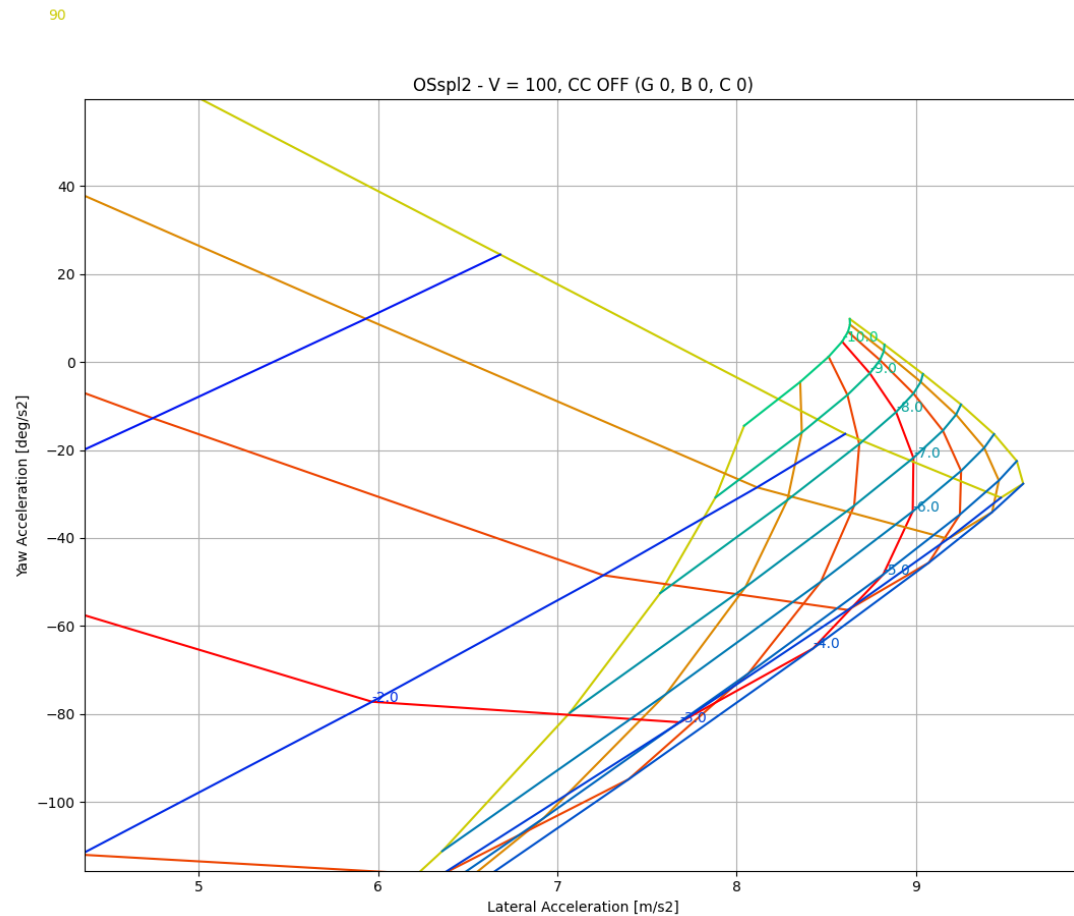


The vehicle can reach up to its lateral acceleration limit in a stable configuration. Let's look at the YMD at the same speed (100km/h).



Again, the vehicle “comes back” after its lateral acceleration limit with a “ripple” in the right part of

the diagram. By following the red line (SWA 0) between -9 and -10 deg, we see that its slope is negative, satisfying the dN/day stability criterion.



This is happening because the front tire is approaching saturation, while the rear is well past it, and it is now operating just after its friction peak, with a negative local derivative.

Data from the “linearized snapshot” of the model in this point are found below:

Variable	Value	Unit
m	1000	kg
J	2560	kg*m ²
l_F	1.25	m
l_R	1.25	m
C_F	-2416	N/rad
C_R	-12738	N/rad
V_x	27.78	m/s
s	-1.835	m
C	-10322	N/rad
$mV_x^2 + Cs$	790547	Nm
Φ	-1.48e10	

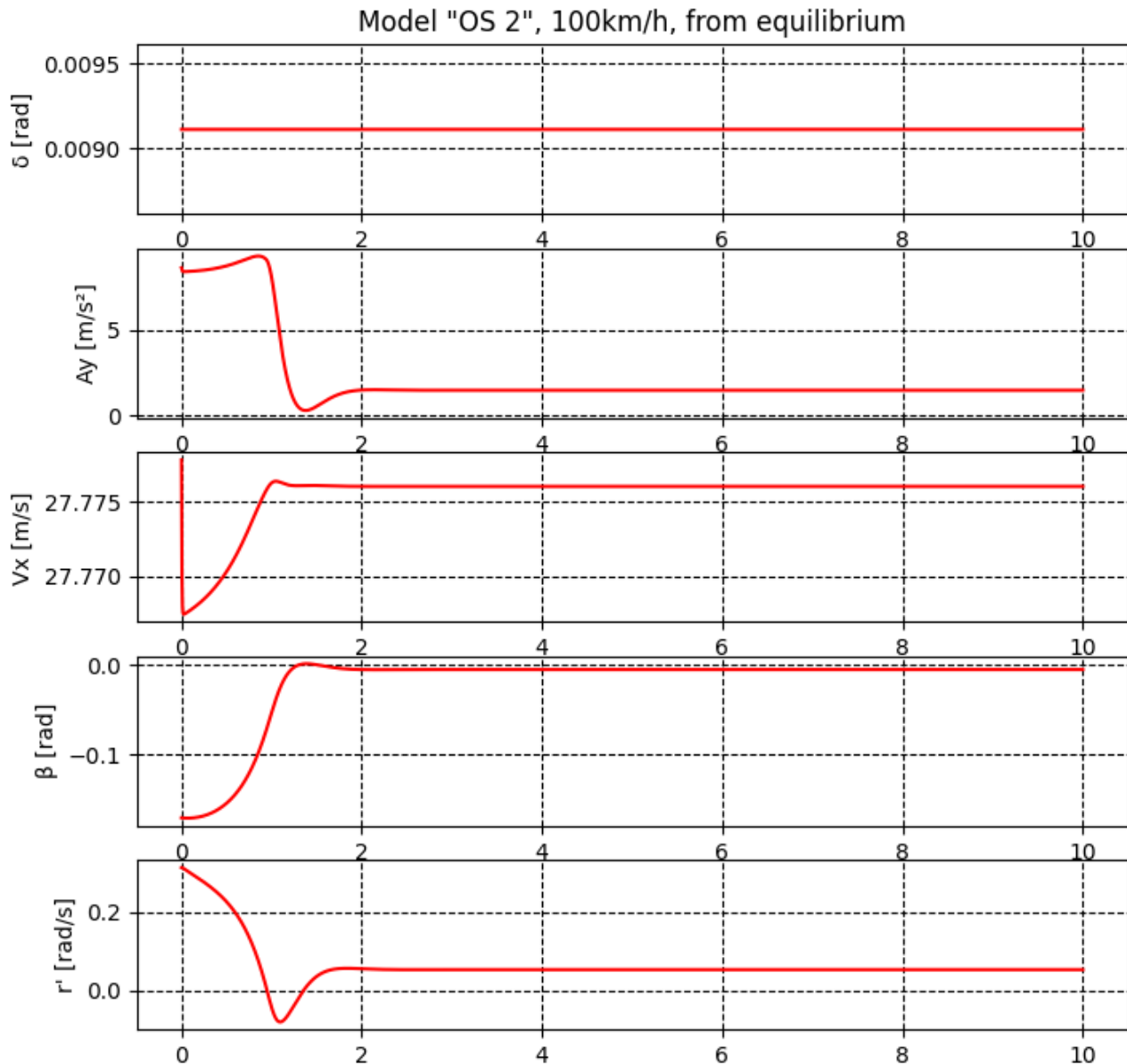
By this effect, the sum of cornering stiffnesses C is negative; at the same time, due to the negative product between C_f and C_r , the Delta of the control theory equation is becoming negative, thus providing the two conditions to trick the criterion.

Naturally, both OptimumG's method and the control theory one correctly flag the vehicle as unstable. In particular, root loci are:

$$\lambda_1 = (-4.91+0j)$$

$$\lambda_2 = (6.11+0j)$$

Note that both roots are real, with no oscillating part. We can see this in a time-integration example:



Which again, moves away (and fast) from its initial equilibrium, falling back, in this case, to another equilibrium point. This might suggest that the vehicle *is* stable in this configuration (it reaches a stable trim), and I'm tempted to think that if "at least a_y is decreasing", the end result might be more stable than with $dN/d\beta$, but I have no proof to this and I'm quite skeptical about it.

Conclusions

At least the following cases cause some errors in the most widespread methods to interpretate a yaw moment diagram:

Case	Linear Stability	OptimumG	Stability Index
Low V_x , $s < 0$, high C	OK	False Negative	OK
$C < 0$, $\Phi > 0$	OK	False Positive	OK
$C < 0$, $\Phi < 0$	OK	OK	False Positive

There seems to be no rigorous way to extract stability information, in its most general form, from a YMD obtained in constant-speed conditions. Due to the inability to make assumptions about C, the dN/da_y criterion may still fail in some corner cases; the $dN/d\beta$ one seems equally fragile (and for sure it's not "the definition of stability"). Unsurprisingly, the Linear Stability criterion provides the most trustworthy results; however, there is no way to "graphically extract" the same information from a YMD.

The supposed strength of the YMD is its ability to plot, at once, a variety of operating conditions, with some simple criteria for its interpretation. As stated in [3], "*It is important to mention that the notions of control and stability can be applied not only to the centre, but to any point in the yaw moment diagram*": it does not appear to be the case.

If such simple criteria do not *always* work, and with no clue about *when* they might not work, one should complement its information with something else (Linear Stability, speed/C,s assumptions). Which kind of defeats the point of using it.

I think the YMD is a bit the "Alfa Romeo of Vehicle Dynamics tools". It's beautiful, unreliable, it breaks if you look at it too close. But don't worry, you can still be a real Vehicle Dynamics Engineer without ever buying one.

Thanks

I'd like to thank William Oberlies and Andrea Quintarelli for reading through this and giving me their feedback. Also, thank you to Guillem Navarrete, Oriol Sala and my colleagues whom I discussed this with, and who have repeatedly been kind enough not to change topic during the many times I brought this up at lunch break.

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- [3] Optimum G, "The four secrets for Chassis Happiness," November 2017. [Online]. Available: <https://students.optimumg.com/the-four-secrets-for-chassis-happiness/>. [Accessed 2026 August 3].